Teacher notes
Topic A

Rolling without slipping

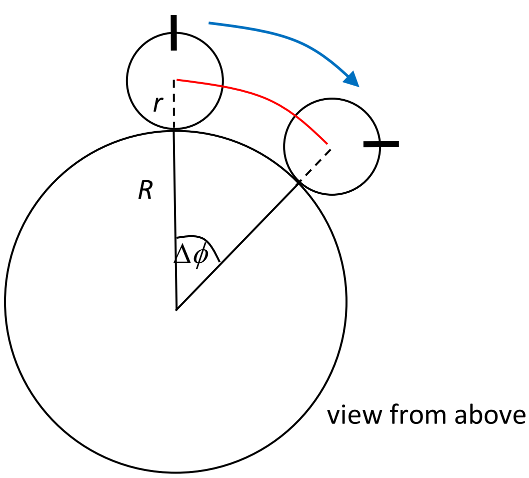
Consider a horizontal large ring of radius *R* and a smaller ring of radius *r*. Suppose that $\frac{R}{r}=3$. The small ring rolls without slipping on the circumference of the larger ring. How many revolutions does the small ring perform when its center of mass returns to its original position?

This was problem #17 in the 1982 SAT exam which was taken by over 300 000 students. The five possible choices given were:

A $\frac{3}{2}$ **B** 3 **C** 6 **D** $\frac{9}{2}$ **E** 9

and do not include the right answer! (The right answer is ***not*** B.)

It is interesting that students with physics knowledge have a distinct advantage in solving this problem. All we must do is use the concept of **rolling without slipping**.



Suppose the ring rolls without slipping so that an angle $\Delta\phi$ is swept. The center of mass of the smaller ring travels a distance $\Delta s=(R+r)\Delta\phi$. Hence $\frac{\Delta s}{\Delta t}=(R+r)\frac{\Delta\phi}{\Delta t}$. But $\frac{\Delta s}{\Delta t}=v$, the linear speed of the center of mass. That linear speed is also given by $v=\omega r=r\dfrac{\Delta\theta}{\Delta t}$, **since we have rolling without slipping**. Here $\Delta\theta$ is the angle by which the small ring rotated about its axis through the center in time Δt.

Hence

$$(R+r)\frac{\Delta\phi}{\Delta t}=r\frac{\Delta\theta}{\Delta t}$$
$$(R+r)\Delta\phi=r\Delta\theta \qquad \text{now divide by } 2\pi$$
$$(R+r)\frac{\Delta\phi}{2\pi}=r\frac{\Delta\theta}{2\pi}$$

When the center of mass of the small ring returns to its original position, $\Delta\phi=2\pi$. The number of rotations *n* of the small ring will be $\dfrac{\Delta\theta}{2\pi}$ and so:

$$R+r=rn$$
$$4r=rn$$
$$n=4$$

In general,

$$R+r=rn$$
$$n=\frac{R}{r}+1$$

You can convince yourself that this answer is correct by taking the limit $R\to 0$. Clearly, it now takes just one revolution of the outer ring to bring itself to its original position, exactly as the formula says.

Most people give the answer $n=3$. Their reasoning is that *n* should be the ratio of the circumferences of the rings: $n=\dfrac{2\pi R}{2\pi r}=\dfrac{R}{r}=3$. The mistake here is that the center of mass of the small ring moves not along the circumference of the larger ring but on the circumference of a circle of radius $R + r$. Hence

$$n=\frac{2\pi(R+r)}{2\pi r}=\frac{R}{r}+1=4\,.$$

If the smaller ring moves along the inner side of the bigger ring, then a similar analysis gives:

$$R-r=rn$$
$$n=\frac{R}{r}-1$$

You may want to watch the Veritasium video https://www.youtube.com/watch?v=FUHkTs-Ipfg for more information on this with convincing demonstrations of what is going on.

For another Physics solution, suppose that the small ring rolls without slipping with constant speed *v*. Then the time for the center of mass of the ring to return to its initial position is $t = \frac{2\pi(R+r)}{v}$. We have rolling without slipping and so $v = r\frac{\Delta\theta}{\Delta t}$. So, in this time the angle swept is $\Delta\theta = \frac{v}{r}\Delta t = \frac{v}{r}\frac{2\pi(R+r)}{v} = \frac{2\pi(R+r)}{r}$, hence the number of revolutions is $n = \frac{\Delta\theta}{2\pi} = \frac{(R+r)}{r} = \frac{R}{r} + 1$.